Mathematics in Image Processing

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- We shall discuss this only in the discrete setting

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- Two issues: (i) Enough information in y to determine x; (ii) How to extract the information y holds about x: Decoder

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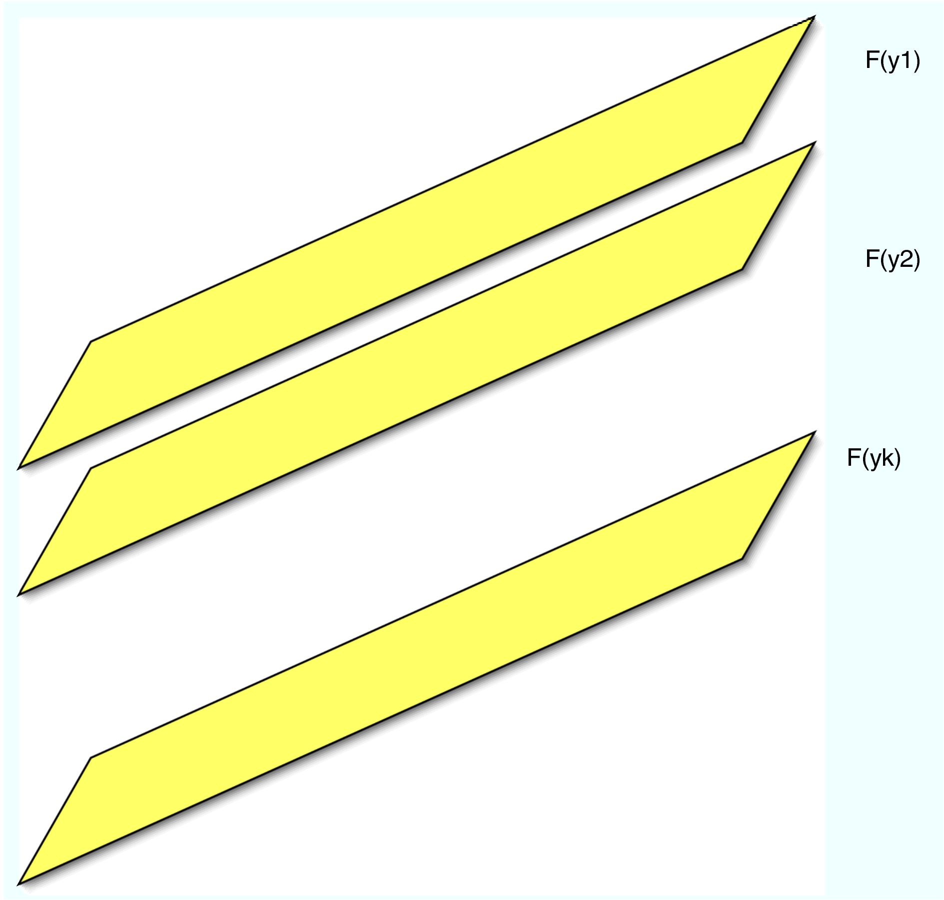
The sets $\mathcal{F}(y)$



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- Note that all $x \in \mathcal{F}(y)$ are approximated by the same $\bar{x}$

Measuring Sparsity

- Compressed Sensing models signals as sparse in some basis
- By linear transformation, we can assume WOLOG that x is sparse with respect to the canonical basis on $\mathbb{R}^N$
- The support of x is $\mathrm{supp}(x) := \{i : \ x_i \neq 0\}$
- $\Sigma_k := \{x : \ \#\mathrm{supp}(x) \leq k\}$
- Note that Σ_k is a union of k dimensional subspaces: $\Sigma_k = \cup_{\#(T)=k} X_T$ where $X_T = \{x : \mathrm{supp}(x) \subset T\}$
- First Question: Given k, N what is the smallest n for which there is (Φ, Δ) such each vector in Σ_k is captured exactly $\Delta(\Phi(x)) = x, \quad x \in \Sigma_k$

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- Answer $n = 2k$

What matrices do the job?

- $\Phi = [v_1, \dots, v_N], \quad v_1, \dots, v_N$ columns of Φ
- We say Φ has the independence property (IP) of order k if all choices of k column vectors are independent
- If $T = \{i_1, \dots, i_m\}$ is a set of column indices
- $\Phi_T = [v_{i_1}, \dots, v_{i_m}]$ is the $n \times \#(T)$ submatrix of Φ formed from these columns
- IP means $\Phi_T^* \Phi_T := (\langle v_i, v_j \rangle)_{i,j \in T}$ is invertible (positive eigenvalues) whenever $\#(T) = k$

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Theorem: If Φ is any $n \times N$ matrix and $2k \leq n$, then the following are equivalent:

(i) There is a Δ such that $\Delta(\Phi(x)) = x$, for all $x \in \Sigma_k$,
(ii) $\Sigma_{2k} \cap \mathcal{N}(\Phi) = \{0\}$,
(iii) the matrix Φ_T has the independence property of order $2k$.

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- Vandermonde matrix. Choose $x_1 < x_2 < \cdots < x_N$

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_N \\ \vdots & \vdots & \cdots & \vdots \\ x_1^{2k-1} & x_2^{2k-1} & \cdots & x_N^{2k-1} \end{pmatrix} \tag{3}$$

Naive Decoding

$$\Delta(y) := \operatorname*{Argmin}_{z\in\Sigma_k} \|y - \Phi(z)\|_{\ell_2^n}$$

- $X_T := \{z : \ \mathrm{supp}(z) \subset T\}$
- $x_T := \operatorname*{Argmin}_{z\in X_T} \|y - \Phi z\|_{\ell_2^n} \rightarrow x_T = [\Phi_T^* \Phi_T]^{-1} \Phi_T y$
- $T^* := \mathrm{Argmin}_{\#(T)=k} \|y - \Phi(x_T)\|_{\ell_2^n}$
- $\Delta(y) := x_{T^*}$

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- We would need this norm controlled for any T of size k

Optimal Stable Systems

- Candes-Romberg-Tao; Donoho: Compressed Sensing
- Two important discoveries: good matrices, decoding
- Restricted Isometry Property (RIP) of order k: There exists $0 < \delta = \delta_k < 1$ such that

$$(1-\delta)\|x\|_{\ell_2^N}^2 \leq \|\Phi(x)\|_{\ell_2^n}^2 \leq (1+\delta)\|x\|_{\ell_2^N}^2, \quad x \in \Sigma_k$$

- Equivalently the eigenvalues of $\Phi_T^*\Phi_T$ are in $[1-\delta, 1+\delta]$
- Decode by ℓ_1 minimization

$$\Delta(y) := \inf_{x \in \mathcal{F}(y)} \|x\|_{\ell_1^N}$$

- Candes-Tao: If Φ satisfies the RIP of order $3k$ then given any $x \in \Sigma_k$ we have $\Delta(\Phi(x)) = x$ for the ℓ_1 minimization decoder. Moreover, the decoding is stable

Building matrices

- How can we build matrices that satisfy RIP for the largest value of k
- Given n, N we can construct such matrices for any $k \leq c_0 n / \log(N/n)$
- The additional $\log(N/n)$ is the price we pay for stability
- This is the largest possible range of k
- How can we construct such Φ?
- We want to create a lot of vectors $v_1, \ldots, v_N$ in $\mathbb{R}^n$ such that any choice of k of them are far from being linearly dependent

Three constructions

- We choose at random N vectors from the unit sphere in $I\!R^n$ and use these as the columns of Φ
- We choose each entry of Φ independently from the Gaussian distribution with mean 0 and variance n^{-1}
- We use Bernouli process and create a matrix with entries $\frac{\pm 1}{\sqrt{n}}$, with equal probability
- With high probability each of these random constructions yields a matrix Φ with RIP of order k for the (largest) range $k \leq c_0 n / \log(N/n)$

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- Probability is only used to prove existence of Φ. The sensing algorithm is constructive (not probabilistic) once we find a Φ.

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- Park City, Utah: July 2009